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A PROOF OF VAUGHT'S CONJECTURE FOR ω-STABLE THEORIES

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Dedicated to the memory of Abraham Robinson on the tenth anniversary of his death

ABSTRACT

In this paper it is proved that if T is a countable complete ω-stable theory in ordinary logic, then T has either continuum many, or at most countably many, non-isomorphic countable models.

Introduction

Vaught's conjecture is the statement that for any countable theory T in $L_{\omega_1\omega}$, T has either $2^{\aleph_0}$, or at most countably many isomorphism types of countable models. Vaught's conjecture is not known to hold even for theories in finitary logic. In this paper, a proof is given for Vaught's conjecture for T a countable ω-stable (totally transcendental) theory.

The results of this paper are due to Shelah. Makkai has written down the proof in the way Harrington patiently explained it to him. Makkai has no part in the work other than checking it and writing it up.

Related earlier results can be found in [1], [2], [3], [5], [6].

The expository paper [M] is used as preliminaries to the present paper. References such as A.2, B.3, etc., refer to [M]. Besides [M], Section 2 of the paper [H–M] is also essential for the present paper; however, if the reader is willing to accept two (crucial) propositions without proof, he can read the present paper without reference to [H–M].

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§1. ENI types, dimension, ENI–NDOP

Throughout the paper, T is a countable complete ω-stable theory.

DEFINITION 1.1. A type (or ideal type) p is an *eventually non-isolated, strongly regular* type (briefly: p is an ENI type) if it is strongly regular (SR; see D.13), and for some finite set B, $p \mid B$ is defined (see Section B) and is non-isolated. p is NENI if it is SR, and it is not ENI.

Note that by the "open mapping theorem" (A.8), if p does not fork over B, and $p \mid B$ is non-isolated, then p is non-isolated. Hence, in the above definition, with B, any finite superset of B will work as well.

To emphasize the elementary character of the following proposition, let us temporarily call a type $p \in S(A)$ eni if A is finite, and there is a finite $B \supset A$ such that all nf extensions of p to B are non-isolated. In other words, p is ENI iff it is eni and SR. Note that in the following proposition, if, in addition, p is SR, then $|I| = \dim(p, M)$.

PROPOSITION 1.2. *Let A be a finite set, $p \in S(A)$, $A \subset M$, I a maximal A-independent set of realizations of p in M.*

(i) *If I is finite, then p is eni.*

(ii) *If M is prime over some finite set, and p is eni, then I is finite.*

(iii) *If p is eni, then there is finite $B \supset A$ such that all nf extensions of p to B are non-isolated, and in addition, $t(\vec{B}/A)$ is isolated.*

PROOF. *ad(i)*. By the defining property of I, no nf extension of p to $A \cup I$ is realized in M; thus, all nf extensions of p to $A \cup I$ are non-isolated. Thus, if I is finite, then (with $B = A \cup I$) p is eni.

ad(ii). Suppose M is prime over the finite set $A' \supset A$, and that I is infinite. Let B be any finite set containing A'. We will show that p has an isolated nf extension to B; by the open mapping theorem, this will imply that p is not eni as desired. Let N be a model prime over B. Since M is prime over A', there is an elementary A'-isomorphism $f : N \xrightarrow{\sim} N'$ such that $M \subset N'$; let $B' = f(B)$. Let I^0 be a finite subset of I such that $B' \perp_{AI^0} I$; let $c \in I - I^0$; it follows that $c \perp_A B'$, in other words $t(c/B')$ is a nf extension of p. Since $c \in M \subset N'$, and N' is prime over B', $t(c/B')$ is isolated. We have found an isolated nf extension of p to B'; since $\vec{B} \equiv \vec{B}'(A)$, there is one for B as well.

ad(iii). Suppose that p is eni. Let M be a model prime over A, and let I be a maximal independent /A set of realizations of p in M. By (ii), I is finite. Since no nf extension of p to $A \cup I$ is realized in M, all such extensions must be non-isolated. Of course, for $B = A \cup I$, $\vec{B}$ has an isolated type over A since $B \subset M$. □

PROPOSITION 1.3. *For SR types, the property of being ENI is invariant under the equivalence relation* $\not\perp$. *If* p_1, p_2 *are SR,* $p_1 \not\perp p_2$, *and* p_1 *is ENI, then* p_2 *is ENI.*

PROOF. Equivalently, we show that if p_1 is ENI, p_2 is NENI, then $p_1 \perp p_2$. Let A be a large enough finite set such that $p_1 | A$ is non-isolated, and $p_2 | A$ is defined. Without loss of generality, $p_1 = p_1 | A$ and $p_2 = p_2 | A$. Let M be a model prime over A. By 1.2(i), $\dim(p_2, M) = \omega$, and of course, $\dim(p_1, M) = 0$. The latter fact and D.21(ii) give us $p_1^\omega \overset{w}{\perp} M/A$ (now $I = \emptyset$). Since $\dim(p_2, M) = \omega$, p_2^ω is realized in M; therefore $p_1^\omega \overset{w}{\perp} p_2^\omega$. By C.7(i), we conclude that $p_1 \perp p_2$, as promised. □

The following proposition is due to Bouscaren and Lascar [2]. For the sake of completeness, we include the proof of it.

PROPOSITION 1.4. *Suppose that* $p \in S(B)$ *is SR,* $A \subset B$, B *finite, and* $p \not\perp A$. *Then for any* B' *with* $\vec{B}' \overset{s}{\equiv} \vec{B}(A)$, *and any* M *containing* A, B *and* B', $\dim(p_{B'}, M) = \dim(p, M)$.

PROOF. We start by two easy remarks. If p, q are SR types over a model N, $M \supset N$, and $p \not\perp q$, then $\dim(p, M) = \dim(q, M)$. Namely, if $\langle a_i \rangle_{i \in I}$ is a p-basis for M, for each $i \in I$ we can find $b_i \in M$ realizing q such that $a_i \not\downarrow_N b_i$ (see D.18); since the a_i, b_i have weight 1 over N, it is easy to see that $\langle b_i \rangle_{i \in I}$ is independent over N; this shows $\dim(p, M) \leqq \dim(q, M)$, and the other inequality is symmetric. Secondly, if $p \in S(M)$ is of weight 1, and $N = M(a)$ with a finite tuple a, then $\dim(p, N)$ is finite; if we had an infinite p-basis I for N, there would be $b \in I$ such that $b \downarrow_M a$ (any $b \in I - I_0$ with a finite $I_0 \subset I$ such that $a \downarrow_{MI_0} I$), contradicting $b \in M(a) - M$ and the open mapping theorem A.8 (the last two facts imply that $b \not\downarrow_M a$).

Turning to the proof of the proposition, let us note that without loss of generality, $A = \emptyset$ (pass to a new theory $T' = \mathrm{Th}(\mathfrak{G}, a)_{a \in A}$). First assume in addition to all the hypotheses of the proposition, that $B' \downarrow B$. Let $M_0 \subset M$ be a model prime over $B \cup B'$. Notice that $\vec{B} \frown \vec{B}' \equiv \vec{B}' \frown \vec{B}$; this follows from the finite equivalence relation theorem (B.3), $B' \downarrow B$, and $\vec{B}' \overset{s}{\equiv} \vec{B}$. It follows that there is an automorphism of M_0 that maps B into B' and B' into B. This clearly shows that, for $p' = p_{B'}$, $\dim(p, M_0) = \dim(p', M_0)$. Since $p \not\perp \emptyset$ and $B' \downarrow B$, we have $p \not\perp p'$ by C.6(i). It follows by the first of the above two remarks that $\dim(p \mid M_0, M) = \dim(p' \mid M_0, M)$. Hence, by the addition formula, D.21(iii), $\dim(p, M) = \dim(p, M_0) + \dim(p \mid M_0, M) = \dim(p', M_0) + \dim(p' \mid M_0, M) = \dim(p', M)$, as required.

Finally, we turn to the general case. Let $\vec{B}''$ be such that $\vec{B}'' \overset{s}{\equiv} \vec{B}' \overset{s}{\equiv} \vec{B}$, and $BB' \mathrel{\unicode{x2AEB}} B''$; let $p'' = p_{\vec{B}''}$, and let $N = M(B'')$. By the case handled above, $\dim(p, N) = \dim(p'', N) = \dim(p', N)$. Also, since $p \not\perp p''$, $p' \not\perp p''$ (and thus $p \not\perp p'$), we have $\dim(p \mid M, N) = \dim(p' \mid M, N)$ by our first remark; moreover, this dimension is finite by our second remark. Since $\dim(p, N) = \dim(p, M) + \dim(p \mid M, N) = \dim(p', M) + \dim(p' \mid M, N)$ $(= \dim(p', N))$, it follows that $\dim(p, M) = \dim(p', M)$, as desired. □

LEMMA 1.5. *Suppose p is stationary and has weight* 1, *$p \not\perp B$, $p \perp A$ and $B \mathrel{\unicode{x2AEB}}_A C$. Then $p \perp C$.*

PROOF. Find an a-model M containing B such that $M \mathrel{\unicode{x2AEB}}_A C$. Since $p \not\perp M$, there is a weight 1 type $q \in S(M)$ such that $p \not\perp q$ (see D.11(v)). Since $p \perp A$, we have $q \perp A$ (see D.5′). By C.8, $q \perp C$; by D.5′ again, $p \perp C$. □

LEMMA 1.6. *Suppose B finite, $q \in S(B)$ a stationary type of weight* 1, *and $q \perp A$. Then there are: an a-model M containing A, an element c, and an a-model $M[c]$ a-prime over Mc, such that c/M is SR, $B \subset M[c]$, and $q \perp M$.*

PROOF. Choose an a-model M_0 containing A, such that $B \mathrel{\unicode{x2AEB}}_A M_0$. By C.8, $q \perp M_0$. Let $N = M_0[B]$. Let $\vec{c} = \langle c_i \rangle_{i<n}$ be a SR basis for N over M_0, i.e., $\vec{c}$ is a maximal M_0-independent system of elements in N, each having a SR type over M_0; among others $N = M_0[\vec{c}]$ (see especially D.15). If we put $M_{k+1} = M_k[c_k]$ $(k < n)$, then it is easily seen that M_n is a-prime over $M_0\vec{c}$, and more generally, over $M_k\langle c_i \rangle_{k \leq i < n}$. Therefore, without loss of generality, $M_0 \subset M_1 \subset \cdots \subset M_n = N$. Since $q \perp M_0$, and $q \not\perp M_n$, there is $k < n$ such that, for $M' = M_k$, $M'' = M_{k+1}$,

$$(1) \qquad q \perp M', \qquad q \not\perp M''.$$

Let $c = c_k$. We claim that there is an a-model M a-prime over $M'\langle c_i \rangle_{k<i<n}$ contained in N such that $N = M[c]$. In fact, let $\hat{M}$ be an a-model a-prime over $D \underset{\text{df}}{=} M'\langle c_i \rangle_{k<i<n}$. Then $\hat{M}[c]$ is a-prime over Ac. Note that also, N is a-prime over Dc. Hence, there is an elementary isomorphism over Dc mapping $\hat{M}[c]$ onto N; this maps $\hat{M}$ onto the desired model M. With M', M'', M so defined, we have $M'' \mathrel{\unicode{x2AEB}}_{M'} M$. Since $w(q) = 1$, (1) implies that $q \perp M$ (see 1.5). Since $B \subset N = M[c]$, the lemma is proved. □

We consider a-models M_0, M_1, M_2 and M satisfying the following conditions

$$(2)\qquad \begin{cases} M_0 \subset M_1, \quad M_0 \subset M_2 \\ M_1 \perp\!\!\!\perp_{M_0} M_2 \\ M\ a\text{-prime over } M_1 \cup M_2 \end{cases}$$

Definition 1.7. T has ENI-NDOP if for all a-models M_0, M_1, M_2 and M satisfying (2), whenever $p \in S(M)$ is an ENI type, then either $p \not\perp M_1$, or $p \not\perp M_2$.

Remark. Compare Section 1 in [H–M].

Proposition 1.8. (i) *T having ENI-NDOP is equivalent to saying that whenever we have* (2), *and p is an ENI type, then $p \not\perp M$ iff $p \not\perp M_1$ or $p \not\perp M_2$.*

(ii) *If T has ENI-NDOP, then whenever $M_0, M_1, \cdots, M_n$ are a-models such that*

$$(3)\qquad \begin{cases} M_0 \subset M_i \qquad (1 \leqq i \leqq n), \\ \langle M_i \rangle_{1\leqq i \leqq n} \text{ is independent over } M_0, \\ M \text{ is } a\text{-prime over } \bigcup_{1\leqq i\leqq n} M_i, \end{cases}$$

and p is an ENI type, then $p \not\perp M$ implies $p \not\perp M_i$ for some i, $1 \leqq i \leqq n$.

(iii) *If T has ENI-DOP (the negation of ENI-NDOP), then there are: a finite tuple a, finite tuples d_1, d_2 extending a, a finite tuple b extending d_1 and d_2, and an ENI type p over b such that d_1/a, d_2/a are of weight 1, $d_1 \perp\!\!\!\perp_a d_2$, b is dominated by d_1d_2/a, by d_2/d_1, and by d_1/d_2, moreover, $p \perp d_1$ and $p \perp d_2$. Furthermore, we can arrange that $d_1 \equiv d_2(a)$, or $d_1/a \perp d_2/a$.*

Proof. The proofs are identical to corresponding proofs in [H–M], with the important addition of Proposition 1.3. In particular, (i) is proved like 1.2 in [H–M], by using 1.3 (and also D.19). The proof of (ii) is done by arguments used in the proof of 1.4 in [H–M], and (iii) is proved just as 1.5 in [H–M]. □

The last two propositions in this section are somewhat of an afterthought to [M].

Proposition 1.9. *Let M_0 be a countable model, $\mathcal{P}$ a set of SR types over M_0. Then there is a countable model M extending M_0 such that* $\dim(p, M) = 0$ *for all $p \in \mathcal{P}$, and* $\dim(q, M)$ *is infinite for all stationary q with* $\operatorname{dom}(q)$ *a finite subset of M such that $q \perp p$ for all $p \in \mathcal{P}$.*

Proof. The proof is identical to that of D.12′. □

Let us call $p \in S(A)$ *nearly SR via q*, if p is stationary, $q \in S(A)$ is SR, and for some a^0 realizing p, and some b^0 realizing q, we have that a is dominated by b^0/A, b is dominated by a^0/A, and the types $t(a^0/Ab^0)$ and $t(b^0/Aa^0)$ are isolated. Clearly, in this case p has weight 1 in particular.

Suppose $p \in S(A)$ is nearly SR via q, and J is a q-basis for M. Then, since $t(a^0/Ab^0)$ (with a^0 and b^0 as above) is isolated, we can find, for each $b \in J$, an a in M with $ab \equiv a^0b^0(a)$. Taking one such a for each b in J, we form a set I; by C.11(iv), I is an independent set over A. Moreover, given any independent set I' of realizations of p in M, the reverse process always gives rise to an independent set J' of realizations of q. It follows that, for I obtained from J as described, if J is a q-basis for M, then I is a p-basis for M, and conversely. Now we can easily prove the following versions of D.21(ii).

Proposition 1.10. *Suppose $p \in S(A)$ is nearly SR, $A \subset M$ and I is a p-basis for M. Then*

$$p \restriction AI \vdash p \restriction M.$$

Proof. Let p be nearly SR via q. Let a realize $p \restriction AI$; in other words, $I \cup \{a\}$ is an independent set of realizations of p. Let us form J, a q-basis for M, out of I as described above, and in fact, let b be a realization of q such that $ab \equiv a^0b^0(A)$ (with a^0, b^0 as above). Thus b realizes $q \restriction AJ$. Since, by D.21(i), $q \restriction AJ \vdash q \restriction M$, b realizes $q \restriction M$. Since a is dominated by b/A, a realizes $p \restriction M$, completing the proof. □

§2. Consequences of having few countable models

From now on, for the rest of the paper we assume that T has less than $2^{\aleph_0}$ isomorphism types of countable models.

At the referee's suggestion, we give a few introductory words trying to illuminate the rather technical contents of this section. Since at some critical points (notably, the proofs of 2.2 and 2.3), we will inevitably have to rely on the paper [H–M], we feel free to make that paper the point of departure for the purposes of this introduction as well.

The analysis of the models of the theory T in [H–M] relies on the concept of a *concrete chain*: a finite sequence $N_0 \subset \cdots \subset N_n$ of models such that N_0 is a-prime over $\emptyset$, and $N_{k+1} = N_k[a_k]$ for some a_k SR over N_k such that $a_k/N_k \perp N_{k-1}$ for $0 < k < n$. (We deliberately use here the a-variant of the notion described in detail at the beginning of section 5 of [H–M]; this variant is the more basic one, and it is the one that goes into the proof of theorem 5.4 there.) The isomorphism types, called simply chains, of concrete chains, in an appropriate sense, form a tree in a natural way; the (foundation-)rank of this tree is an important invariant, called the depth, of the theory.

Now, the first main point is that here we are interested, essentially, only in chains for which, using the above notation for chains, the top type, $t(a_{n-1}/N_{n-1})$, is ENI. The reason ultimately is that the dimensions of NENI types over finite sets in a countable model cannot matter, since they must necessarily be equal to $\aleph_0$. Call such a chain an ENI chain (being ENI is invariant under isomorphism of concrete chains).

It is a crucial fact that, under the standing hypothesis of this section of T having less than $2^{\aleph_0}$ non-isomorphic countable models, we can show that all ENI chains are of length at most 2 (i.e., n, in the above notation, is at most 2). This is an equivalent way of stating what Proposition 2.4 below says.

We may define a notion of ENI-depth of any t.t. T (or, even a more general T), although it is unclear how useful this notion is. We take the subset of the tree of chains consisting of all ENI chains; closing this set down under taking initial segments results in a subtree; the rank of this subtree might be called the ENI-depth. Of course, the above statement is equivalent to saying that, under the hypothesis of this section, the ENI depth is at most 2.

Another fact is that, for an ENI chain, a_0/N_0 is necessarily trivial, analogously to a stronger fact shown in [H–M] under a different hypothesis; this fact is equivalent to 2.2 below. This fact is used among others to prove 2.3, ENI-NDOP (defined in Section 1); but also, the triviality of such types goes in a direct way into the "final analysis" of models given in Section 3 in a crucial manner.

The reader will notice that the terminology used here does not appear in the body of this section. Rather, we have to make a finer analysis of types over sets, mostly finite sets. In particular, we have to make a more general statement of the kind of relationship that the type $t(a_{k+1}/M_{k+1})$ bears to the element a_k and the set M_k (here again, we used the above notation for chains; $M_{k+1} = M_k[a_k]$). The resulting notion is that of type p *needing* an element c over a set A (p *needs* c/A); in particular, it will be true that $p = t(a_{k+1}/M_{k+1})$ needs a_k over M_k in the above situation. The notion is such that it is invariant under replacing p by a parallel, in fact, by a non-orthogonal, SR type. The most reasonable general formulation of this notion seems to be this: p, a stationary weight 1 type, *needs* c/A iff $p \perp A$, and for some B dominated by c/A, we have $p \not\perp B$. It turns out that, under the hypothesis of few countable models, and for p an ENI type, 'needing' can be formulated in a simpler way; this simpler formulation will be adopted after Proposition 2.7. In fact, we feel that talking about needing does not really help much in understanding the technicalities of the first (larger) half of this section; that is why we do not use the terminology up to 2.8.

The last two results of this section are finiteness results whose significance for a structural description will likely be accepted easily. It should, however, be born in mind that in the "final analysis" not only these results, but others in this section as well, play important roles.

We first repeat a definition from [H–M].

DEFINITION 2.1. A stationary type q is called *trivial* if every nf extension q' of q satisfies the following: whenever I is a set of elements realizing q' such that any two-element subset of I is independent over $\operatorname{dom} q'$, then I is independent over $\operatorname{dom} q'$.

PROPOSITION 2.2. *Let A be a set, c a tuple of weight* 1 *over A, c/A stationary, p an ENI type such that $p \perp A$ and* $\operatorname{dom}(p)$ *is dominated by c/A. Then c/A is trivial.*

PROOF. The proof is essentially identical to that of 2.2 in [H–M]. The proof remains valid upon replacing λ by $\aleph_0$, and changing all references to a-models to ordinary models, in particular, a-prime models to prime models. The reason is that p, the type 'supported by' c/A, is not only of weight 1, but it is ENI. This latter fact is used to show at the appropriate place that $\dim(p_d, M_0) < \aleph_0$. In fact, for M_1, the prime model over D, we have $\dim(p_d, M_1) < \aleph_0$ by 1.2(ii), since p is ENI. The equality $\dim(p, M_0) = \dim(p, M_1)$ follows by the same (easy) argument as the one used in the proof of D.12″(iii), since we have $p_d \perp D'$. □

PROPOSITION 2.3. *T has ENI-NDOP.*

PROOF. Again, the proof is essentially the same as that of 2.3 in [H–M]; the changes to be made are the same ones as in the case of 2.2. Of course, the "reduction" 1.5 in [H–M] is replaced by 1.8(iii). □

PROPOSITION 2.4. *Suppose p is an ENI type, C/A is stationary, and has weight* 1, *$p \not\perp C$ and $p \perp A$. Then $C/A \not\perp \varnothing$.*

PROOF. Without loss of generality, C is finite, $C = c_0$. Suppose, on the contrary, that there are p_0, c_0 and A such that p_0 is ENI, c_0/A has weight 1, $p_0 \not\perp c_0$, $p_0 \perp A$ and $c_0/A \perp \varnothing$. We first *claim* that then there are finite tuples $a \subset b \subset c$ and an ENI type p with $\operatorname{dom}(p) = c$ such that c/b, b/a are stationary, and of weight 1, $p \perp b$, and $c/b \perp a$. By applying 1.6 to $q = t(c_0/A)$, we can find an a-model M, b_0 SR over M, such that $A \subset M[b_0]$, $c_0 \downarrow_A M[b_0]$, and $c_0/M[b_0] \perp M$. By 1.5, $p_0 \perp M[b_0]$. Since $p_0 \not\perp c_0$, by D.19 and 1.3 there is an ENI type p over $(M[b_0])[c_0]$ such that $p \not\perp p_0$. Let c_1 be a finite tuple in $M[b_0][c_0]$ containing c_0 such that p is based on c_1, let b_1 be a finite tuple in $M[b_0]$ containing b_0 such that $c_1/M[b_0]$ is based on b_1, and let a be a finite tuple in M such that c_1b_1/M is based on a. Let $b = ab_1$, $c = ab_1c_1$. Then $c/M[b_0]$ is based on b; by C.12(ii), c is dominated by $c_0/M[b_0]$, hence by D.2(ix), $c/M[b_0]$ is of weight 1, and so by D.2(iv), c/b is of weight 1. Similarly, b/a is of weight 1. Since $p_0 \perp M[b_0]$, and $p \not\perp p_0$, we have $p \perp b$; since $c/M[b_0] \perp M$, we have $c/b \perp a$. We have proved the *claim*.

Now, using the data of the claim, we proceed to construct $2^{\aleph_0}$ pairwise non-isomorphic countable models, in contradiction to our assumption on T. Without loss of generality, a is the empty tuple (namely, having proved the existence of many models under the above conditions with $a = \emptyset$, we can apply the result to $T' = \mathrm{Th}(\mathfrak{C}, a)$; but the conclusion for T' implies that for the original T).

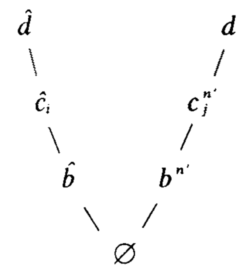
Let X be an arbitrary subset of $\omega - \{0\}$. For $n \in X$, let b^n be a tuple such that $b^n \equiv b$, and $\langle c_i^n \rangle_{i<n}$ be a system independent over b^n of elements c_i^n such that $c_i^n b^n \equiv cb$. Let $C_n = \{c_i^n : i < n\}$. We furthermore make sure that $\langle C_n \rangle_{n \in X}$ is independent. Let $p_i^n = p_{c_i^n}$. We *claim* that there is a model M_X such that $\dim(p_i^n, M_X)$ is finite for all $n \in X$, $i < n$ and $\dim(q, M_X)$ is infinite for all stationary q with a finite $\mathrm{dom}(q)$ included in M_X such that $q \perp p_i^n$ for all $n \in X$, $i < n$.

Let $n \in X$ and $i < n$ be fixed, and let M be a model prime over C_n such that $M \downarrow_{b^n} C_{X-\{n\}}$. Since $p_i^n \perp b^n$, we have $p_i^n \mid M \perp C_{X-\{n\}}/M$, from which it follows that in $M(C_{X-\{n\}})$, $p_i^n \mid M$ is not realized (any element d of $M(C_{X-\{n\}}) - M$ satisfies $d \not\downarrow_M C_{X-\{n\}}$). On the other hand, by 1.2(ii), $\dim(p_i^n, M)$ is finite. It follows (by D.21(iii)) that $\dim(p_i^n, M(C_{X-\{n\}}))$ is finite, hence for M_0, the prime model over C_X, $\dim(p_i^n, M_0)$ is finite (since M_0 is embedded in $M(C_{X-\{n\}})$). Since this is true for all $n \in X$ and $i < n$, the existence of a suitable M_X follows from 1.9 applied to $\mathcal{P} = \{p_i^n \mid M_0 : n \in X, i < n\}$.

By a *class* we now mean an equivalence class of elements $\hat{b}$ in M_X such that $\hat{b} \equiv b$, under the equivalence relation $(\ \ ) \not\downarrow (\ \ )$. The class containing $\hat{b}$ is denoted $[\hat{b}]$. Given a class $\mathcal{B}$, and $\hat{b} \in \mathcal{B}$, let $m(\hat{b})$ be the supremum of all cardinals $m \leqq \omega$ such that there is a system $\langle \hat{c}_i \rangle_{i<m}$, independent over $\hat{b}$, of elements $\hat{c}_i$ in M_X such that $\hat{c}_i\hat{b} \equiv cb$ and such that for all $i < m$, $\dim(p_{\hat{c}_i}, M_X)$ is finite, and let $m(\mathcal{B}) = \sup\{m(\hat{b}) : \hat{b} \in \mathcal{B}\}$. We *claim* that $m([b^n]) = n$, and that $m(\mathcal{B}) = 0$ for a class $\mathcal{B}$ distinct from any $[b^n]$, $n \in X$. This claim will clearly imply that for distinct $X, X' \subset \omega - \{0\}$, $M_X \neq M_{X'}$, since X is recovered from M_X as the set $\{m(\mathscr{B}) : \mathscr{B}$ a class in M_X, $m(\mathscr{B}) \neq 0\}$.

To show the first assertion of the claim, we note that $m([b^n]) \geqq n$ holds directly by the construction. On the other hand, let $\hat{b} \in [b^n]$, $\langle \hat{c}_i \rangle_{i<m}$ independent over $\hat{b}$, $\hat{c}_i\hat{b} \equiv cb$, let $\hat{p}_i = p_{\hat{c}_i}$, and assume that $\dim(\hat{p}_i, M)$ is finite for all $i < m$.

Note that $\hat{p}_i \perp p_j^{n'}$ for all $n' \in X - \{n\}$, $j < n'$; this follows from 3.3 in [H–M], since the "tree"



with $\hat{d}$ realizing $\hat{p}_i$, d realizing $p_j^{n'}$ is a normal tree; also, the assertion can be shown by an elementary computation using given (non-)orthogonalities and $\hat{b} \mathop{\smile\!\!\!\!\!\!\mid}\, b_j^{n'}$. Therefore, it follows by the defining property of M_X, and the fact that $\dim(\hat{p}_i, M)$ is finite, that for every $i < m$ there is $\hat{\imath} < n$ such that $\hat{p}_i \not\perp p_{\hat{\imath}}^n$. Now, we notice that $\hat{p}_i \perp \hat{p}_{i'}$ for $i \neq i'$, i, $i' < m$; this is because $\hat{c}_i \mathop{\smile\!\!\!\!\!\!\mid}_{\hat{b}} \hat{c}_{i'}$, and $\hat{p}_i \perp \hat{b}$. Therefore, $i \neq i'$ implies $\hat{\imath} \neq \hat{\imath}'$; this proves that $m \leqq n$, as promised.

If $\mathscr{B} = [\hat{b}] \neq [b^n]$ for all $n \in X$, i.e., $\hat{b} \mathop{\smile\!\!\!\!\!\!\mid}\, b^n$ for all $n \in X$, then if $\hat{c}\hat{b} \equiv cb$, and $\hat{p} = p_{\hat{c}}$, we have $\hat{p} \perp p_j^n$ for all $n \in X$, $j < n$, just as in the previous paragraph. By the defining property of M_X, we conclude that $m(\mathscr{B}) = 0$, as required for the second assertion of the claim.

The proof is complete. □

PROPOSITION 2.5. *Suppose p is an ENI type, $p \not\perp B$, and B is dominated by C/A. Then $p \not\perp AC$.*

PROOF. Without loss of generality, B is a finite set. Assume, contrary to the assertion, that $p \perp AC$. Choose an a-model $M \supset A$ such that $BC \mathop{\smile\!\!\!\!\!\!\mid}_A M$, and an a-model $M' \supset M$ such that $B \mathop{\smile\!\!\!\!\!\!\mid}_{MC} M'$. It follows that we have $b \mathop{\smile\!\!\!\!\!\!\mid}_A M'$, hence by 1.5 and the assumptions, $p \perp M'$. Let $\vec{b} = \langle b_i \rangle_{i<n}$ be a sequence of elements realizing SR types over M' such that $M'[B] = M'[\vec{b}]$. By a suitable choice of the models $M'[b_i]$, we have that $M'[B]$ is a-prime over $\bigcup_{i<n} M'[b_i]$. Hence, by ENI-NDOP (1.8(ii)), and $p \not\perp B$, there is $i < n$ such that $p \not\perp N$ for $N = M'[b_i]$. Since $p \perp M'$, by 2.4 it follows that $b_i / M' \not\perp \varnothing$. Hence, there is $q \in S(M)$, an SR type, such that $b_i / M' \not\perp q$. By D.18, there is $b \in N$ realizing $q \mid M'$ such that

$N = M'[b]$. We *claim* that there is a copy of $M[b]$ such that N is a-prime over $M' \cup M[b]$. Indeed, let $M[b]$ be any copy of the model a-prime over Mb, and let N' be the model a-prime over $M' \cup M[b]$. Then, N' is a-prime over $M'b$; to see this, we use $M' \downarrow_M b$, and $M' \downarrow_M M[b]$ as a consequence. Therefore, there is an isomorphism mapping N' onto N over the set $M'b$. The image of $M[b]$ under this isomorphism is the desired copy of $M[b]$. Having $M[b]$ in the desired way, by ENI-NDOP again, and by $p \perp M'$, we conclude that $p \not\perp M[b]$. Now, note again that $c \downarrow_M b$; since B is dominated by c/M, $B \downarrow_M b$, and hence $B \downarrow_M M[b]$. Since $p \not\perp B$, and $p \not\perp M[b]$, by 1.5 we obtain $p \not\perp M$, a contradiction to $p \perp M'$. □

PROPOSITION 2.6. *Let p be an ENI type, suppose c/A, c'/A are stationary and weight* 1, $c \not\downarrow_A c'$, $p \not\perp Ac$, *and* $p \perp A$. *Then* $p \not\perp Ac'$.

PROOF. With B a finite set such that p is based on B, let M be an a-model containing A such that $Bcc' \downarrow_A M$. Then $p \perp M$. Note that we have $c \not\downarrow_M c'$. Let $\vec{c} = \langle c_i \rangle_{i<n}$ be a sequence independent over M of elements c_i of weight 1 over M such that $N \underset{\text{df}}{=} M[Bcc'] = M[\vec{c}]$ and such that $c_0 = c$. Since $c \not\downarrow_M c'$, $\vec{c}' = c'{}^\frown \langle c_i \rangle_{1 \leqq i < n}$ also is a maximal independent system of elements having weight 1 over M, hence $N = M[\vec{c}']$. Now, since $p \not\perp M[c]$, $p \perp M$, and $M[c] \downarrow_M c_i$ $(1 \leqq i \leqq n)$, by 1.5 we have $p \perp Mc_i$ for $1 \leqq i < n$. Since $N = M[\vec{c}']$, there are copies $M[c']$, $M[c_i]$ $(1 \leqq i < n)$ of the models a-prime over Mc' and Mc_i, respectively, such that N is a-prime over $M[c'] \cup \bigcup_{1 \leqq i < n} M[c_i]$. Since $p \perp Mc_i$, and $M[c_i]$ is dominated by c_i/M, by 2.5, $p \perp M[c_i]$. Therefore, by ENI-NDOP (1.8(ii)), necessarily $p \not\perp M[c']$, and by 2.5 again, $p \not\perp Mc'$. Since we have $B \downarrow_{Ac'} Mc'$, and $p \not\perp B$, we have $p \not\perp Ac'$ as required. □

PROPOSITION 2.7. *Suppose p is an ENI type*, $A \subset A'$, $c \downarrow_A A'$, $p \not\perp A'c$, $p \perp A'$. *Then* $p \not\perp Ac$.

PROOF. Without loss of generality, A is an a-model M. (To see this, choose an a-model M such that $A \subset M$ and $M \downarrow_A A'c$; in particular, $c \downarrow_M A'$ and $c \downarrow_{A'} M$. From $p \not\perp A'c$ and $p \perp A'$, it follows that $p \perp MA'$ by 1.5. Thus, we have the hypotheses with M for A and MA' for A'. From $p \not\perp Mc$, $p \not\perp A'c$ and $Mc \downarrow_{Ac} A'c$, $p \not\perp Ac$ follows by 1.5.) Let $N = [M[A'], M[c]]$, the a-prime model over the union of $M[A']$ and $M[c]$. Since $p \not\perp N$, and $M[A'] \downarrow_M M[c]$, by ENI-NDOP we have either $p \not\perp M[A']$ or $p \not\perp M[c]$. But the first possibility is ruled out, since by 2.5 it would imply $p \not\perp MA'$, contrary to $p \perp A'$ $(M \subset A')$. Thus, $p \not\perp M[c]$, and by 2.5 again, $p \not\perp Mc$ as desired. □

We now introduce some terminology. We say that the ENI type p *needs* c/A if

c/A is stationary and of weight 1, $p \not\perp Ac$ and $p \perp A$. Notice that 1.6 together with 2.5 says that $p \perp A$ implies that p needs c/A' with some $A' \supset A$ and some c, with c/A' SR. Also, 2.6 says that if p needs c/A, and $c \not\downarrow_A c'$, c'/A is stationary and of weight 1, then p needs c'/A. 2.7 says that if c/A is stationary, $A \subset A'$, $c \downarrow_A A'$, and p needs c/A', then c needs c/A. In all the above, p is assumed to be ENI. We call a type $q \in S(A)$ *supportive* if for some (any) c realizing q, there is an ENI type p needing c/A. Note that 2.4 says that every supportive type is $\not\perp \emptyset$. Also, it is clear that every supportive type is $\not\perp$ to a SR supportive type (*exercise*).

PROPOSITION 2.8. *Suppose p, p' are ENI types, p needs c/b, p' needs c'/b'. Suppose furthermore that $c/b \perp c'/b'$. Then $p \perp p'$.*

PROOF. Start by choosing tuples $\hat{b}$ and $\hat{b}'$ such that $\hat{b} \stackrel{s}{\equiv} b$, $\hat{b}' \stackrel{s}{\equiv} b'$, and

$$bcb'c',\ \hat{b},\ \hat{b}' \text{ are independent (over } \emptyset). \tag{3}$$

Since $q = t(c/b) \not\perp \emptyset$ (q is a supportive type), we have $q_{\hat{b}} \not\perp q$ (see C.6(i)). Let M be an a-model containing $b\hat{b}\hat{b}'$ such that $c \downarrow_{b\hat{b}\hat{b}'} M$; then $c \downarrow_b M$ by (3). Let $\hat{c}$ realizing $q_{\hat{b}} \restriction M$ be such that $c \not\downarrow_M \hat{c}$. Since p needs c/b, we have (by $c \downarrow_b M$) that p needs c/M. Hence, by $c \not\downarrow_M \hat{c}$, p needs $\hat{c}/M$. Since $\hat{c} \downarrow_{\hat{b}} M$ ($\hat{c}$ realizes $q_{\hat{b}} \restriction M$), p needs $\hat{c}/\hat{b}$. Note also that we have $\hat{c} \downarrow_{\hat{b}} \hat{b}'$ (since $\hat{b}'$ belongs to M).

By a symmetric argument, we can find $\hat{c}'$ realizing $q'_{\hat{b}'}$ (for $q' = t(c'/b')$) such that p' needs $\hat{c}'/\hat{b}'$, and $\hat{c}' \downarrow_{\hat{b}'} \hat{b}$. Now, we apply the fact that $q \perp q'$. Since $q_{\hat{b}} \not\perp q$, $q'_{\hat{b}'} \not\perp q'$, and all the types involved are stationary and of weight 1, it follows that $q_{\hat{b}} \perp q'_{\hat{b}'}$, hence $q_{\hat{b}} \restriction \hat{b}\hat{b}' \perp q'_{\hat{b}'} \restriction \hat{b}\hat{b}'$. Since $\hat{c}$ realizes the first, $\hat{c}'$ the second, of the last two types, we conclude that we have

$$\hat{c} \underset{\hat{b}\hat{b}'}{\downarrow} \hat{c}'. \tag{4}$$

Since p needs $\hat{c}/\hat{b}$, and $\hat{c} \downarrow_{\hat{b}} \hat{b}'$, p needs $\hat{c}/\hat{b}\hat{b}'$. Similarly, p' needs $\hat{c}'/\hat{b}\hat{b}'$. By (4) and 1.5, we have $p \perp \hat{b}\hat{b}'\hat{c}'$. Since $p' \not\perp \hat{b}\hat{b}'\hat{c}'$, it follows that $p \perp p'$. □

PROPOSITION 2.9. *There are only finitely many equivalence classes of supportive types under the equivalence relation $\not\perp$.*

PROOF. Suppose, on the contrary, that there are infinitely many pairwise $\perp$ supportive types. We'll construct $2^{\aleph_0}$ non-isomorphic countable models.

Suppose $q \in S(b)$ is a supportive type. Notice that there are finitely many strong types over $\emptyset$ extending $r = t(b)$, and for any b', b'' realizing r, if $b' \stackrel{s}{\equiv} b''$, then $q_{b'} \not\perp q_{b''}$ (since $q \not\perp \emptyset$). Hence there are altogether finitely many $\not\perp$-classes that contain an isomorphic copy $q_{b'}$ of q ($b' \equiv b$).

Suppose further that q' is another supportive type. If $q' \not\perp q$, and $\hat{q}$ is an isomorphic copy of q, then for $\hat{q}'$, the isomorphic copy of q' under the automorphism of $\mathfrak{C}$ that maps q into $\hat{q}$, we clearly have $\hat{q}' \not\perp \hat{q}$. In other words, if q, q' are two supportive types (in fact, if q, q' are stationary weight-1 types), then the set of equivalence classes under $\not\perp$ of isomorphic copies of q is either identical to, or disjoint from, the set of equivalence classes under $\not\perp$ of isomorphic copies of q'.

It now follows that there is a sequence $\langle q_n \rangle_{n<\omega}$ of supportive types such that for $n, n' < \omega$, $n \neq n'$, every isomorphic copy of q_n is $\perp$ to every isomorphic copy of $q_{n'}$.

Let $b_n = \operatorname{dom}(q_n)$, let c_n be a realization of q_n; we may assume that b_n is a subtuple of c_n; let p_n be an ENI type needing c_n / b_n; let $d_n = \operatorname{dom}(p_n)$; we may assume that c_n is a subtuple of d_n, and also that p_n is not isolated.

Let X be an arbitrary subset of ω. For $n \in X$, let $\hat{b}_n, \hat{c}_n, \hat{d}_n$ be such that $\hat{b}_n \equiv b_n$, $\hat{b}_n\hat{c}_n\hat{d}_n \equiv b_n c_n d_n$, and let us ensure that the system pictured by

$$\begin{array}{ccccc} \cdots & \hat{d}_n & \cdots & \hat{d}_{n'} & \cdots \\ & | & & | & \\ \cdots & \hat{c}_n & \cdots & \hat{c}_{n'} & \cdots \\ & | & & | & \\ \cdots & \hat{b}_n & \cdots & \hat{b}_{n'} & \cdots \end{array} \qquad (n, n' < \omega)$$

is independent relative to the obvious "height-3" partial ordering (see A.11, A.12). Let $\hat{p}_n = (p_n)_{\hat{d}_n}$. With M_0 the prime model over $\hat{d}_X$ ($= \{\hat{d}_n : n \in X\}$), $\dim(\hat{p}_n, M_0) = 0$. The reason is that since $\hat{p}_n \perp \emptyset$, and $\hat{d}_n \mathop{\downarrow} \hat{d}_{X-n}$, we have $\hat{p}_n \vdash \hat{p}_n \mid d_X$, hence $\dim(\hat{p}_n, M_0) = \dim(\hat{p}_n \mid \hat{d}_X, M_0)$; the latter is 0 since $\hat{p}_n \mid \hat{d}_X$ is non-isolated (since $\hat{p}_n$ is non-isolated), and M_0 is prime over $\hat{d}_X$. By 1.9, there is a countable model $M = M_X$ extending M_0 such that $\dim(p_n, M) = 0$ for all $n \in X$, and $\dim(p, M)$ is infinite for any SR type p with $\operatorname{dom}(p)$ a finite subset of M, and $p \perp \hat{p}_n$ for all $n \in X$.

We *claim* that $n \in X$ iff there are $\tilde{b}, \tilde{c}, \tilde{d}$ in M such that $\tilde{b}\tilde{c}\tilde{d} \equiv b_n c_n d_n$ and $\dim((p_n)_{\tilde{d}}, M)$ is finite. Indeed, the 'only if' direction is true by construction (take $\tilde{b} = \hat{b}_n$, $\tilde{c} = \hat{c}_n$, $\tilde{d} = \hat{d}_n$). Conversely, assume $n \in \omega$, $\tilde{b}\tilde{c}\tilde{d}$ is in M, and has the same type as $b_n c_n d_n$. For $n' \neq n$, $\tilde{c}/\tilde{b} \perp \hat{c}_{n'}/\hat{b}_{n'}$, by the choice of the types q_n. Since $\tilde{p} = (p_n)_{\tilde{d}}$ needs $\tilde{c}/\tilde{b}$, and $\hat{p}_{n'}$ needs $\hat{c}_{n'}/\hat{b}_{n'}$, by 2.8 we have that $\tilde{p} \perp \hat{p}_{n'}$. Thus, if $n \notin X$, then $\tilde{p}$ is $\perp$ to all $\hat{p}_{n'}$ for $n' \in X$, hence by the defining property of M, $\dim(\tilde{p}, M)$ is infinite; this shows the *claim*.

The claim clearly shows that $X \neq X'$ implies $M_X \not\cong M_{X'}$, which proves the proposition. □

PROPOSITION 2.10. *Let A be any finite set. There are only finitely many equivalence classes of ENI types $\not\perp$ to A under the equivalence relation $\not\perp$.*

PROOF. Without loss of generality, $A = \emptyset$. Let M_0 be a fixed copy of the prime model of T. By D.19, and 1.3, every ENI type $\not\perp$ to $\emptyset$ is $\not\perp$ to one over M_0, hence one over a finite tuple in M_0. Suppose the assertion of the proposition fails. Using also an argument used in the proof of 2.9, we find that there is a sequence $\langle p_n \rangle_{n<\omega}$ of ENI types such that $b_n \underset{\text{df}}{=} \operatorname{dom}(p_n)$ is in M_0, and every isomorphic copy of p_n is $\perp$ to every isomorphic copy of $p_{n'}$, for $n \neq n'$. Let X be an arbitrary subset of ω. Since $\dim(p_n, M_0)$ is finite (see 1.2(ii)), for all $n < \omega$, by 1.9 there is a model M_X extending M_0 such that $\dim(p_n, M_X)$ is finite for all $n \in X$, and $\dim(p, M_X)$ is infinite for every stationary weight-1 type p with $\operatorname{dom}(p)$ a finite subset of M_X such that $p \perp p_n$ for all $n \in X$.

X can be recovered from M_X as follows: $n \in X$ iff there is an isomorphic copy p of p_n, with $\operatorname{dom}(p)$ in M_X, such that $\dim(p, M_X)$ is finite. Certainly, the 'only if' direction is clear. Supposing that p is an isomorphic copy of p_n, we have that $p \perp p_{n'}$ for all $n' \neq n$. Hence, if $n \notin X$, then $\dim(p, M_X)$ is infinite, by the defining property of M_X. This shows what we want. □

§3. The final analysis

In this section, we state and prove a theorem that amounts to a characterisation up to isomorphism of an arbitrary countable model in terms of certain invariants. The statement of the characterisation takes some preparation. Once the characterisation is stated, however, the Vaught conjecture for T is seen to be an essentially trivial consequence of it. Namely, using the characterisation, one can write down a Scott sentence σ_M of any countable model of T (σ_M is a sentence of $L_{\omega_1\omega}$ such that the countable models of σ_M are exactly those isomorphic to M), and one sees that the quantifier rank of σ_M is low, in particular, $\operatorname{qr}(\sigma_M) < \omega \cdot \omega$. It is well-known that from this the Vaught conjecture for T follows (see [4]). The writing out of σ_M will be left to the reader as a trivial but tedious exercise.

First, we set up the "reference-points" of the invariants.

Let M be a fixed copy of the prime model.

Let Q be a finite set of pairwise $\perp$ SR supportive types over M such that every supportive type is $\not\perp$ to a member of Q. Since every supportive type q is $\not\perp$ to the empty set (2.4), and since q is supportive, q' SR and $q \not\perp q'$ imply that q' is supportive (2.6), every supportive type is $\not\perp$ to a supportive type over M (by using D.19). Also, there are only finitely many $\not\perp$-classes of supportive types (2.9). It follows that Q with the stated properties exists.

Choose and fix a realization c_q of each $q \in Q$; put $C = \{c_q : q \in Q\}$.

Let A be a finite subset of M such that every $q \in Q$ is based on A.

For each $q \in Q$, let $M_q \supset M$ be a fixed copy of $M(c_q)$; we write M_c for M_q if $c = c_q$.

For each $q \in Q$, let P_q be a finite set of pairwise $\perp$ ENI types over M_q such that every ENI type which is $\not\perp$ to Ac_q is $\not\perp$ to a member of P_q. P_q exists by 2.10. We also write P_c for P_q with $c = c_q$.

For each $c \in C$, let B_c be a finite subset of M_c such that $Ac \subset B_c$, every $p \in P_c$ is based on B_c, and $p \mid B_c$ is non-isolated (since p is ENI, by 1.2(iii) such B_c can be chosen).

Let A' be a finite subset of M such that $A \subset A'$, each $t(\vec{B}_c M)$ $(c \in C)$ is based on A', and each $t(\vec{B}_c / A'c)$ is isolated.

For each $q \in Q$, let $P'_q \,(= P'_{c_q})$ be the subset of P_q consisting of those members of P_q that are $\perp$ to A'. Thus, every member of P'_c needs c/A'. Moreover, if p is any ENI type that needs c/A', we have that p needs c/A by 2.7, since $c \downarrow_A A'$, hence p is $\not\perp$ to some member p' of P_c, and p' must belong to P'_c.

Let P be a finite set of pairwise $\perp$ ENI types, each $\not\perp$ to A', each over M, each $\perp$ to every member of Q, such that every ENI type which is $\not\perp$ to A, is $\not\perp$ to either a member of O, or a member of P. P exists by 2.10.

Let B be a finite subset of M such that $A' \subset B$, every member of P is based on B, and such that for $p \in P$, $p \mid B$ is non-isolated.

Let A'' be a finite set such that $A' \subset A'' \subset \mathrm{acl}(A')$ (with the algebraic closure meant in $\mathfrak{C}^{eq}$) such that $\vec{B}/A''$ is stationary, and $\vec{B}_c/A''$ is stationary for every $c \in C$. By B.4 and by the fact that the theory is t.t., it is easy to see that such A'' exists.

Let us put:

$$Q' = \{q \mid A'' : q \in Q\},$$

$$\left.\begin{aligned} B'_c &= B_c \cup A'', \\ P''_c &= \{p \mid B'_c : p \in P'_c\}, \end{aligned}\right\} \quad c \in C$$

$$B' = B \cup A'',$$

$$P' = \{p \mid B' : p \in P\}.$$

Primed (or double-primed) items are updated versions of the corresponding previous items; these latter can be discarded. Also, let B^0 be the set of all the finite tuples $\vec{B}'_c$ for $c \in C$, $B^0 = \{\vec{B}'_c : c \in C\}$. We can discard the c's in C, and work with the b's in B^0. Also, let R^0 be the set of all types $t(b/A'')$ for $b \in B^0$; thus, $B^0 = \{b_r : r \in R^0\}$ for a unique assignment $r \mapsto b_r$ such that b_r realizes r. Let us write a^0 for $\vec{A}''$, d^0 for $\vec{B}'$, P^0 for P', and P^0_b for P''_c if $b = \vec{B}'_c$; also, $P^0_r = P^0_{b_r}$ for $r \in R^0$.

Besides the items marked with superscript 0, we do not need the others above, and we may use the notation used for them for other things. Here is the list of the properties of the items we'll need.

a^0 is a subtuple of d^0, $t(d^0/\emptyset)$ is isolated, and $t(d^0/a^0)$ is stationary.

R^0 is a set of pairwise $\perp$, stationary weight-1 types over a^0 such that every supportive type is $\not\perp$ to a member of R^0.

(The fact that $t(b_c/a)$ is of weight 1 follows from the fact that B'_c is dominated by c over A'', and that c/A'' is of weight 1, see D.2(ix). The same reasons, together with the fact that the elements of Q are pairwise $\perp$, are sufficient for the elements of R^0 to be pairwise $\perp$, as is easily seen.)

If $r \in R^0$, M is any model containing a^0, I is an r-basis in M, then $r \mid a^0 I \vdash r \mid M$. (This follows from 1.10 since clearly r is nearly SR.)

Every type in R^0 is trivial (this follows from 2.2).

B^0 is a set of representative realizations of the types in R^0; $B^0 = \{b_r : r \in R^0\}$, b_r realizes r. a^0 is a subtuple of each b in B^0.

For each $b \in B^0$, P^0_b is a finite set of pairwise $\perp$ non-isolated SR types over b, each $\perp$ to a^0, such that any ENI type that needs b/a^0 is $\not\perp$ to a member of P^0_b. (Indeed, if P is an ENI type that needs b/a^0, then, for c, the 'c-part' of b, p needs c/a^0 (this follows from 2.5, and the fact that b is dominated by c/a^0); hence, by the above, p is $\not\perp$ to a member of P^0_b.)

P^0 is a finite set of pairwise $\perp$ non-isolated SR types over d^0, each $\not\perp$ to a^0, each $\perp$ to every member of R^0, such that every ENI type $\not\perp$ to a^0 is $\not\perp$ either to a member of R^0, or to a member of P^0. (Note that passing from A' to A'' ($= a^0$) did not affect the essential properties of A', since $A' \subset A'' \subset \mathrm{acl}(A')$.)

This completes the list of "reference-points" for the invariants.

Now, let M be an arbitrary countable model. We choose a in M such that $a \equiv a^0$, and d in M such that $ad \equiv a^0 d^0$. We consider the types p_d for all $p \in P^0$; p_d is the copy of p under the change of d^0 to d. Let $\vec{m} = \langle \dim(p_d, M) : p \in P^0 \rangle$ be the vector of dimensions of the types p_d in M. Note that by 1.4, and by the fact that $t(d/a)$ is stationary, $d \equiv d'(a)$ implies that $\vec{m}$ computed with d and that computed with d' are equal; in other words, $\vec{m}$ depends only on (M and) a.

Next, we consider the types r_a for $r \in R^0$; in other words, if b realizes r_a, we have $ba \equiv b_r a^0$. For $r \in R^0$, we let X_r be the family of all $\not\downarrow_a$-classes of realizations of r_a; X_r is the set of all classes of the form $\{b' \in M : b' \not\downarrow_a b,\ b' \models r_a\}$ for all b in M realizing r_a. Put $X = \bigcup_{r \in R^0} X_r$.

Now let $\mathscr{B}$ be a class in X_r $(r \in R^0)$. Pick $b \in \mathscr{B}$; in particular, b realizes r. Consider the types p_b for $p \in P_r^0$ and let $\vec{n} = \langle n_p \rangle_{p \in P_r^0}$ be the vector of dimensions $n_p \underset{\text{df}}{=} \dim((p)_b, M)$. Of course, $\vec{n}$ depends on b; $\vec{n} = \vec{n}(b)$. With a fixed $\mathscr{B}$ in X_r, we consider the set of all possible dimension-vectors $D(\mathscr{B}) = \{\vec{n}(b) : b \in \mathscr{B}\}$. For $r \in R^0$, let us denote by $\mathscr{D}_r^0$ the set of all countable sets of vectors $\vec{n} = \langle n_p \rangle_{p \in P_r^0}$, with arbitrary $n_p \leqq \omega$; $\mathscr{D}_r^0$ is a set defined independently of M. Clearly, $D(\mathscr{B}) \in \mathscr{D}_r^0$ for $\mathscr{B} \in X_r$. We count how many times each $D \in \mathscr{D}_r^0$ occurs as $D(\mathscr{B})$ in M for $\mathscr{B} \in X_r$; this number is denoted by s_D. In other words, s_D is the cardinality of the set of all $\mathscr{B} \in X_r$ such that $D(\mathscr{B}) = D$. We have that $s_D \leqq \aleph_0$, and $s_D = 0$ except for countably many $D \in \mathscr{D}_r^0$.

The system of invariants for (M, a) consists, by definition, of the vector $\vec{m}$, and the vectors

$$\langle s_D \rangle_{D \in \mathscr{D}_r^0}$$

for all $r \in R^0$. (Again, notice that once the copy a of a^0 has been chosen in M, the above data are all uniquely defined.)

THEOREM 3.1. *Suppose T is a countable ω-stable theory with less than $2^{\aleph_0}$ many non-isomorphic countable models. Let $a \equiv a' \equiv a^0$, $a \in M$, $a' \in M'$, M and M' are countable models. Then, if (M, a) and (M', a') have the same invariants, they are isomorphic.*

The rest of the section is devoted to the proof of the theorem. As we indicated above, the Vaught conjecture for T is an essentially trivial consequence of the theorem.

Let M be a countable model. Let us make all the specifications and choices as above relative to M. Let R be the set of all types $(r)_a$, for $r \in R^0$. For $r \in R$, let B_r be a maximal independent set of elements of M each realizing r, and let $B = \bigcup_{r \in R} B_r$. Note that B is a maximal independent set of elements of M each realizing a type in R.

Let N be a *maximal atomic* model over aB in M: $aB \subset N \subset M$, N atomic over aB, and if $N \subset D \subset M$, D atomic over aB, then $N = D$. Since the union of an increasing chain of sets atomic over aB is again atomic over aB, and for every $D \subset M$ atomic over aB there is N', $D \subset N' \subset M$, N' atomic over aB, such N clearly exists

For $b \in B_r$, $r \in R$, let P_b be the set of all the types $p_b \mid N$, for $p \in P_r^0$. Let P_1 be the set of all the types $p_a \mid N$ for $p \in P^0$. Put $P = \bigcup_{b \in B} P_b \cup P_1$. Finally, let I be a P-basis for M, i.e. a maximal independent set of realizations of types in the set P.

CLAIM 3.2. *M is atomic over $N \cup I$.*

PROOF. Let M' be maximal atomic over $N \cup I$ in M. We are going to show that $M' = M$.

First we show (*Subclaim* 1) that no NENI type $p \in S(N)$ is realized in M. Indeed, suppose $p \in S(N)$ is NENI, and p is realized in M by c, say. Then $p \perp r$ for every $r \in R$; otherwise $r \mid N$ is realized in M, contradicting the maximality of B. We now show that for arbitrarily large finite subsets D of N (for every finite subset D_0 of N there is a finite D, $D_0 \subset D \subset N$) such that $t(c/aBD)$ is isolated; this will show that the set $N \cup \{c\}$ is atomic over aB, contradicting the maximality of N.

Let D be any finite subset of N such that $c \downarrow_D N$, $a \subset D$. Find finite $B' \subset B$ such that $D \downarrow_{B'} B$. Since B is an independent set over a, and we have $B - B' \downarrow_a DB'$, we have that $B - B'$ is an independent set over DB', and for each $b \in B - B'$, $t(b/DB')$ is parallel to $t(b/a)$, an element of R. Since p is orthogonal to every member of R, $t(c/DB')$ (a type parallel to p) is orthogonal to $B - B'/DB'$. It follows that

$$(1) \qquad p \mid DB' \vdash p \mid DB' \cup (B - B') = p \mid DB.$$

Since p is NENI, and $D \cup B'$ is finite, $p \mid DB'$ is isolated. By (1), $p \mid DB$ is isolated as well. Since this holds for arbitrarily large finite $D \subset N$, we obtain a contradiction as said. Subclaim 1 is established.

Next, we show that (*Subclaim* 2), in fact, no NENI type over M' is realized in M. The proof is similar to that of Subclaim 1. Suppose $p \in S(M')$ is NENI, and is realized by c in M. Let $D \subset M'$ be any finite set such that p is based on D, let I' be a finite subset of I such that $D \downarrow_{NI'} I$, and let D' be a finite subset of N such that $a \in D'$ and $DI' \downarrow_{D'} N$. By the maximality of I, p is orthogonal to every type in the set P. Therefore, it follows as (1) does above that

$$(2) \qquad p \mid NDI' \vdash p \mid NDI.$$

We have that $p \perp N$; otherwise there is a SR type $p' \in S(N)$ such that $p \not\perp p'$.(see D.19); by 1.3, p' is NENI, and by D.18, p' (in fact, even $p' \mid M'$) is realized in M, contradicting Subclaim 1. Hence, we have $p \perp D'$. It follows that $p \perp N/D'$, and by $DI' \downarrow_{D'} N$ that $p \perp N/D'DI'$, hence

$$(3) \qquad p \mid D'DI' \vdash p \mid NDI'.$$

(2) and (3) give $p \mid D'DI' \vdash p \mid NDI$. Since p is NENI, $p \mid D'DI'$ is isolated; therefore, so is $p \mid NDI$, and we obtain a contradiction to the maximality of M' as above, completing the proof of Subclaim 2.

Subclaim 3 is the statement that every ENI type over M is $\not\perp$ to an element of $P \cup R$.

Let $p \in S(M)$ be an ENI type. If $p \not\perp a$, then, by choice of the set $P_1 \subset P$, p is $\not\perp$ to a member of $P_1 \cup R$. So, we may assume that $p \perp a$.

By 1.6, there is a set A containing a, and an element $\hat{d}$ such that $\hat{d}/A$ has weight 1, and p needs $\hat{d}/A$. Also, A can be chosen "sufficiently large", e.g. we may assume A is an a-model. The type $\hat{d}/A$ thus shown to be a supportive type, we have that $\hat{d}/A$ is $\not\perp$ to a member r of R (this is because of the choice of R^0, and the fact that the property of R^0 in question is invariant under automorphisms of $\mathfrak{C}$, hence it is shared by R). Find $\hat{b}$ realizing $r \mid A$ such that $\hat{b} \not\downarrow_A \hat{d}$ (recall that A is sufficiently large). By 2.6, and since p is ENI, we have that p needs $\hat{b}/A$. By 2.7, it follows that p needs $\hat{b}/a$. We have $p \not\perp a\hat{b}$ and $p \perp a$, hence necessarily, $M \not\downarrow_a \hat{b}$. We *claim* that $B_r \not\downarrow_a \hat{b}$. Indeed, if we had $B_r \downarrow_a \hat{b}$, then $\hat{b}$ would realize $r \mid aB_r$; hence, by the version of D.21(ii) for r, stated above, $\hat{b}$ realizes $r \mid M$, i.e. $M \downarrow_a \hat{b}$, which is false, showing the *claim*. Since r is trivial (see above), and $B_r \not\downarrow_a \hat{b}$, there is $b \in B_r$ such that $b \not\downarrow_a \hat{b}$. Since p needs $\hat{b}/a$, it follows that p needs b/a. By choice of the set $P_b \subset P$, we have that $p \not\perp$ to some member of P_b, completing the proof of Subclaim 3.

To complete the proof of 3.2, let us assume that $M - M' \neq \emptyset$. Then there is $c \in M - M'$ such that c/M' is SR. By Subclaim 2, $p = t(c/M')$ is ENI. By Subclaim 3, $p \mid M$, hence p itself, is $\not\perp$ either to a member of R, or to a member of P. The first possibility contradicts the maximality of B, the second the maximality of I. □

Let $\hat{p} = (p^0)_b$ for some $r \in R^0$, $b \in B_r$, $p^0 \in P^0_r$.

CLAIM 3.3. $\dim(\hat{p}, N) = 0$.

PROOF. $b = \operatorname{dom}(\hat{p})$ is an element of B. We have $b \downarrow_a B - \{b\}$. By C.8, and since $\hat{p} \perp a$, we have that $\hat{p} \perp a(B - \{b\})$, hence $\hat{p} \vdash \hat{p} \mid ab(B - \{b\}) = \hat{p} \mid aB$. $\hat{p} \mid aB$ is non-isolated (since $\hat{p}$ is), and since N is atomic over aB, $\hat{p} \mid aB$ is not realized in N. It follows that $\hat{p}$ is not realized in N either. □

Next, consider a type $\hat{p} = (p^0)_d$, $p^0 \in P^0$.

CLAIM 3.4. *$\dim(\hat{p}, N)$ is finite.*

PROOF. Choose a finite subset B' of B such that $B \downarrow_{aB'} d$. Let N' be a model contained in N, and prime over dB'. Since p is a non-isolated SR type, $\dim(\hat{p}, N')$ is finite (see 1.2(ii)). On the other hand,

$$\dim(\hat{p}, N) = \dim(\hat{p}, N') + \dim(\hat{p} \mid N', N);$$

we'll show that the second term of the sum is 0, which will complete the proof.

Recall that $\hat{p}$ is $\perp$ to every member of R. We have $B \downarrow_{aB'} d$ and $B - B' \downarrow_a B'$, hence $B - B' \downarrow_a dB'$. It follows that $q \underset{\text{df}}{=} t(\overline{B - B'}/dB')$ is a product of types that are nf extensions of members of R. Hence, q is orthogonal to $\hat{p}$, and to $\hat{p} \mid dB'$ as well. It follows that

$$\text{(4)} \qquad \hat{p} \mid dB' \vdash \hat{p} \mid dB' \cup (B - B') = \hat{p} \mid dB.$$

N is atomic over aB, hence over dB as well. Since $\hat{p} \mid dB$ is non-isolated, $\dim(\hat{p} \mid dB, N) = 0$. From the relation (4), it follows that $\dim(\hat{p} \mid dB', N) = 0$. A fortiori, $\dim(\hat{p} \mid N', N) = 0$, as desired. □

Now, we can prove Theorem 3.1.

Let M and M' be two countable models, $a \in M$, $a' \in M'$, $a \equiv a' \equiv a^0$, and assume that (M, a) and (M', a') have the same invariants. Introduce the items referred to in 3.2–3.4 in the model M (but not yet in M'). Next, consider a fixed $r \in R^0$, and the set X'_r of all the equivalence classes $\mathscr{B}'$ of realizations of $r_{a'}$ in M' (just as X_r was defined in M). Since $s_D^{(M)} = s_D^{(M')}$ for every $D \in \mathscr{D}_r^0$, clearly, there is a 1-1 correspondence between the classes $\mathscr{B}$ in X_r, and the classes $\mathscr{B}'$ in X'_r such that if $\mathscr{B}'$ corresponds to $\mathscr{B}$ under this correspondence, then $D^{(M)}(\mathscr{B}) = D^{(M')}(\mathscr{B}')$. Consider any $b \in B_r$ and $\mathscr{B}$, the class in X_r containing b. Let $\vec{n} = \vec{n}(b)$ be the vector of the p_b-dimensions of M, for $p \in P_r^0$. The equality $D^{(M)}(\mathscr{B}) = D^{(M')}(\mathscr{B}')$ (for $\mathscr{B}$ and $\mathscr{B}'$ corresponding classes) implies that there is $b' \in \mathscr{B}'$ such that $\vec{n}^{(M')}(b') = \vec{n}(b)$. Let us choose one such $b' \in \mathscr{B}'$, and let us define a map,

$$b \mapsto b'$$

defined for $b \in B = \bigcup_{r \in R^0} B_r$, with b' determined as described for each $b \in B$. Since the b' are pairwise in distinct classes, they are pairwise independent over a'. Since the types in R^0 are trivial, it follows that the family of all b''s (for $b \in B$) is independent over a'. Since B was an R-basis for M (for $R = \{(r)_a : r \in R^0\}$), and hence B has (exactly) one element in each class $\mathscr{B} \in X$, moreover since the correspondence $\mathscr{B} \mapsto \mathscr{B}'$ is bijective on classes, it follows that $B' = \{b' : b \in B\}$ is an R'-basis for M' ($R' = \{(r)_{a'} : r \in R^0\}$). We also have that the map that assigns a' to a, and b' to b for each $b \in B$ is an elementary isomorphism between the sets $\{a\} \cup B$ and $\{a'\} \cup B'$.

Now, let N' be an (elementary) submodel of M' maximal atomic over $a'B'$. The above elementary isomorphism can be extended to one from N onto N'. Let d' be the tuple in N' which is the image of d under the said isomorphism. The types $p_b \mid N$ in P_b ($p \in \bigcup_{r\in R^0} P_r^0$) correspond to $p_{b'} \mid N'$ under the isomorphism; let $P'_{b'} = \{(p)_{b'} \mid N'; p \in \bigcup_{r\in R^0} P_r^0\}$. The types $p_d \mid N$ in P_1 ($p \in P^0$) correspond to $p_{d'} \mid N'$; let $P'_1 = \{(p)_{d'} \mid N' : p \in P^0\}$. Put $P' = \bigcup_{b'\in B'} P'_{b'} \cup P'_1$.

Notice that each type in P_{b_1} is $\perp$ to every type in P_{b_2}, for $b_1 \neq b_2$, both in B, as well as to every type in P_1. This follows easily from the facts that every type in P_{b_1} is $\perp$ to a, and that every type in P_1 is $\not\perp$ to a. It follows that if we decompose I into the union of the I_b ($b \in B$) and I_1, with $I_b = \{c \in I : c$ realizes some type in $P_b\}$, $I_1 = \{c \in I : c$ realizes some type in $P_1\}$, then I_b is a P_b-basis for M, and I_1 is a P_1-basis for M. Furthermore, the types in a fixed P_b, or in P_1, are pairwise $\perp$ by choice, hence

$$I_p = \{c \in I : c \text{ realizes } p\} \qquad (p \in P_b)$$

is a p-basis for M. Let $p \in P_b$, $\hat{p} = p \mid b$; $\hat{p} = (p^0)_b$. Then, since $\dim(\hat{p}, N) = 0$, by the addition formula D.21(iii) it follows that $\dim(p, M) = \dim(\hat{p}, M)$. Of course, the latter is what we denoted by $n_{p^0}(b)$, and it is the same as the cardinality of I_p. If b' corresponds to b under the specific isomorphism, then $n_{p^0}^{(M')}(b') = n_{p^0}^{(M)}(b)$, by the choice of the correspondence $b \mapsto b'$. Thus, if we let $\hat{p}' = (p^0)_{b'}$, and $p' = \hat{p}' \mid N'$, the type in P' corresponding to p, and $I'_{p'}$ any p'-basis for M', then (because of $\dim(\hat{p}', M') = \dim(p', M')$) we have that $|I'_{p'}| = |I_p|$.

For any $p \in P_1$, $\hat{p} = p \mid d$, $k = \dim(\hat{p}, N)$ is finite by 3.4. If p' corresponds to p, $\hat{p}' = p \mid d'$, then of course $\dim(\hat{p}', N') = \dim(\hat{p}, N) = k$. Also, by assumption, $\dim(\hat{p}, M) = \dim(\hat{p}', M')$ (since $\vec{m}^{(M)} = \vec{m}^{(M')}$). Since

$$\dim(\hat{p}, M) = \dim(\hat{p}, N) + \dim(p, M) = k + \dim(p, M),$$

and similarly for M', it follows that $\dim(p, M) = \dim(p', M')$. So, if $I'_{p'}$ is a p'-basis for M', then $|I'_{p'}| = |I_p|$.

Let $I' = \bigcup_{p'\in P'} I'_{p'}$. Then, clearly, I' is a P'-basis for M'. If we let $c \mapsto c'$ be any bijection of I_p and $I'_{p'}$, for each $p \in P$ and the corresponding p' in P', then these bijections together with the given isomorphism of N onto N' is an isomorphism from $N \cup I$ onto $N' \cup I'$. Applying 3.2 to M' now, M' is prime over $N' \cup I'$, as well as M was prime over $N \cup I$. It follows that the isomorphism of $N \cup I$ onto $N' \cup I'$ can be extended to an isomorphism of M onto M'.

This completes the proof of the theorem.

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